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HYDRAULIC JUMP IN THE SHEAR FLOW OF AN IDEAL INCOMPRESSIBLE FLUID

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This article examines a mathematical model representing a long-wave approximation for the motion of a layer of a turbulent ideal incompressible fluid with a free boundary above a level channel bottom. The equations of motion are represented in the form of a system of conservation laws for mass, total momentum, and vorticity in a mixed Eulerian—Lagrangian coordinate system. This system is then used as the basis for finding discontinuous solutions to integrodifferential equations in the theory of long waves. We studied the properties of relations for a hydraulic jump. It is shown that different types of jumps may occur, depending on the parameters of the incoming flow. It is found that discontinuous relations make it possible to determine the flow parameters behind a jump for any supercritical shear flow ahead of the jump. We consider the example of a steady-state solution dependent on a single parameter, this example illustrating the transition from one type of jump to another.

Several investigations have studied the equations of long waves for a turbulent fluid. The authors of [1, 2] obtained infinite series for the conservation laws, while exact steady-state solutions describing flows with a critical layer were obtained in [3]. Particular solutions of the simple-wave type were constructed in [4, 5]. The conditions under which the equations of motion are hyperbolic were explained in [6].

1. Formulation of the Mathematical Model. We will examine the initial-boundary-value problem

$$\begin{gathered} \rho(u_T + uu_X + vu_Y) + p_X = 0, \\ \varepsilon^2\rho(v_T + uv_X + vv_Y) + p_Y = -\rho g,\ 0 \leq Y \leq h(X, T), \\ u_X + v_Y = 0, \\ h_T + u(X, h, T)h_X = v(X, h, T),\ p(X, h, T) = p_0 = \text{const}, \\ v(X, 0, T) = 0,\ u(X, Y, 0) = u_0(X, Y),\ h(X, 0) = h_0(X), \end{gathered} \tag{1.1}$$

describing the flow of a layer of an ideal incompressible uniform (ρ = const) fluid with a free boundary Y = h(X, T) above a level channel bottom Y = 0. Here

$$\begin{gathered} u^1 = (g_1H_1)^{1/2}u,\ v^1 = (g_1H_1)^{1/2}H_1L_1^{-1}v,\ p^1 = g_1R_1H_1p, \\ \rho^1 = R_1\rho,\ X^1 = L_1X,\ Y^1 = H_1Y,\ T^1 = L_1(g_1H_1)^{-1/2}T \end{gathered}$$

are dimensional components of the velocity vector, pressure, density, the cartesian coordinates in a plane, and time; u, v, p, ρ, X, Y, and T are the corresponding dimensionless quantities; R_1 has the dimension of density; g_1 has the dimension of acceleration (acceleration due to gravity G is connected with the dimensionless constant g by the relation $G = g_1g_2$); H_1 and L_1 are the characteristic vertical and horizontal scales. The initial data u_0, h_0 were determined at $0 \leq Y \leq h_0(X)$, $-\infty < X < \infty$.

An approximate theory of long waves (theory of shallow water) is established with passage to the limit $\varepsilon = H_1L_1^{-1} \to 0$. The conservation equation for the vertical component of momentum in the limit gives the hydrostatic law of pressure distribution in the depth direction:

$$P_Y = -\rho g \quad (p(X, h, T) = p_0). \tag{1.2}$$

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We use the continuity equation and (1.2) to obtain the equations

$$p = \rho g(h - Y) + p_0,\ v = -\int_0^Y u_X dY. \tag{1.3}$$

When $\varepsilon = 0$, we can use these relations to reduce problem (1.1) to a Cauchy problem

$$\begin{gathered} \rho(u_T + uu_X + vu_Y) + \rho g h_X = 0 \quad (0 \leqslant Y \leqslant h(X, T)), \\ h_T + \frac{\partial}{\partial X}\left(\int_0^h u dY\right) = 0, \\ u(X, Y, 0) = u_0(X, Y),\ h(X, 0) = h_0(X) \end{gathered} \tag{1.4}$$

for unknown functions u(X, Y, T), h(X, T). Pressure and the vertical component of velocity are determined by Eqs. (1.3). Equation (1.4) leads us to the vorticity conservation equation:

$$(\omega)_T + u(\omega)_X + v(\omega)_Y = 0 \quad (\omega = u_Y)$$

(in the approximation of long-wave theory, vorticity coincides with $-u_Y$). The equation just given has the following consequence: if $\omega = 0$ at the initial moment of time, then $\omega = 0$ at all T. In the case of irrotational flow, Eq. (1.4) reduces to the classical equations of the theory of shallow water:

$$u_T + uu_X + gh_X = 0,\ h_T + (uh)_X = 0. \tag{1.5}$$

Discontinuous solutions of the system of shallow-water equations model moving hydraulic jumps. To obtain relations linking the parameters of the flow on both sides of the jump, we need to convert Eqs. (1.5) into a system representing the conservation laws for horizontal momentum and mass in the fluid layer:

$$(\rho hu)_T + (\rho hu^2)_X + (2^{-1}\rho g h^2)_X = 0,\ \rho h_T + (\rho uh)_X = 0. \tag{1.6}$$

We use (1.6) to obtain relations for the front of the discontinuity $X = x(t)$:

$$[\rho h(u - D)^2 + 2^{-1}\rho g h^2] = 0,\ [\rho h(u - D)] = 0. \tag{1.7}$$

Here, $[f] = f^+ - f^-$ is the difference between the limiting values of the function f on the line of discontinuity $X = x(t)$; $D = x'(t)$ is the velocity of the front. The sign of $u - D$ determines the direction of the flow relative to the front. The condition for stability of the discontinuous flow reduces to decay of the total energy of the layer upon intersection with the line of discontinuity. One consequence of this condition is the inequality

$$|u - D| > \sqrt{gh}; \tag{1.8a}$$

$$|u - D| < \sqrt{gh}. \tag{1.8b}$$

Inequality (1.8a) must be satisfied for limiting values of the flow parameters ahead of the front, while (1.8b) must be satisfied behind the front (supercritical and subcritical flows relative to the front).

More accurate modeling of actual flows requires that we consider the dependence of the profile of horizontal velocity on the vertical coordinate ($u_Y \neq 0$). An approximate approach employed in hydraulics connects mean flow velocity in the depth direction U with mean-square velocity

$$\alpha U^2(X, T) = h^{-1}\int_0^h u^2 dY \quad \left(\int_0^h u dY = hU\right) \tag{1.9}$$

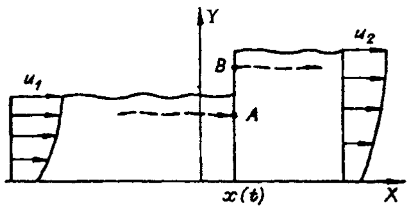

Fig. 1

(α is an empirical parameter which corrects for the momentum of the flow). By integrating the first equation of motion (1.4) over Y and using (1.9), we can close the equations for the mean quantities and use relations for the discontinuity that have an empirical constant.

Our goal here is to propose a model of hydraulic jumps in a turbulent flow on the basis of conservation laws that follow from the basic equations of motion. We propose to use relations for the discontinuity to determine the depth of the layer, mean velocity, and the overall profile of velocity in the depth direction.

We should briefly discuss the main features of the modeling of discontinuous flows such as those being examined here. Discontinuity of the trajectory of the particles is a characteristic feature of the model — a feature that importantly distinguishes hydraulic jumps from other types of discontinuities known in hydrodynamics. The fluid particles suddenly move to another height when they cross the front of the discontinuity (Fig. 1, where the trajectory of a particle is represented by the dashed line). Thus, when deriving the balance relations for the front, we need to connect values of the flow parameters at different Eulerian points (points A and B in Fig. 1). We have no *a priori* knowledge of the location of the point B where the particle leaves the jump region after entering at point A. Thus, the first step in constructing a mathematical model is to change over to mixed Eulerian—Lagrangian coordinates by means of the relations

$$T = t,\ X = x,\ Y = \Phi(x, t, \lambda)$$

(λ is the vertical Lagrangian variable). Here, $\Phi(x, t, \lambda)$ is the solution of the Cauchy problem

$$\Phi_t + u(x, \Phi, t)\Phi_x = v,\ \Phi(x, 0, \lambda) = \lambda h_0(x). \tag{1.10}$$

In the new variables, the flow regions correspond to the zone $0 \leq \lambda \leq 1$, while the equations of motion take the form [6]

$$u_t + uu_x + g\int_0^1 H_x dv = 0; \tag{1.11a}$$

$$H_t + (uH)_x = 0\ (H(x, t, \lambda) = \Phi_\lambda(x, t, \lambda)). \tag{1.11b}$$

A second important element of the modeling process is the selection of the system of conservation equations. Equation (1.11b) expresses the local mass conservation law. After multiplying (1.11a) by H and integrating over λ, we also obtain the conservation law for the total horizontal momentum of the layer:

$$\left(\int_0^1 uHdv\right)_t + \left(\int_0^1 u^2Hdv + 2^{-1}gh^2\right)_x = 0. \tag{1.12}$$

We need one more local relation to close the model. The divergence equation, being a consequence of system (1.4),

$$(\rho Hu)_x + (\rho Hu^2)_x + (pH)_x - (p\Phi_x)_\lambda = 0\ \left(p = \rho g\int_\lambda^1 Hdv + p_0\right) \tag{1.13}$$

formally coincides with the local conservation law for momentum. However, Eq. (1.13) loses meaning for discontinuous solutions of the type we are examining.

In fact, if the quantities u, Φ, H, and p have first-order discontinuities at $x = x(t)$, then the last term in (1.12) becomes meaningless. The quantity under the derivative sign for λ is mathematically indeterminate, this being the product of the discontinuous function p and the function Φ_X. The latter function has a singularity of the Dirac delta function type at $x = x(t)$.

As an additional closing equation, we propose using a corollary of the equations of motion — the vorticity conservation law

$$(\omega H)_t + (u\omega H)_x = 0 \ (\omega = u_Y = u_\lambda H^{-1}), \tag{1.14}$$

which is also formulated in mathematically determinate terms in the case of flows with discontinuous trajectories. In sum, we propose to find discontinuous solutions to the equations of long waves by using system of conservation laws (1.11b), (1.12), and (1.14). Writing these laws in integral form leads us to the following equations at the front of the jump:

$$[H(u - D)] = 0, \ [\int_0^1 H(u - D)^2 d\lambda + 2^{-1}gh^2] = 0, \ [\omega H(u - D)] = 0. \tag{1.15}$$

In the special case of irrotational flow, Eqs. (1.15) coincide with (1.7) (if $u_1 = 0$, then, in accordance with (1.10), $H = h(X, T)$). We use Eqs. (1.15) to obtain the corollaries (the first relation is valid at points on the front where $H(u - D) \neq 0$)

$$[\omega] = 0, \ [\Psi]_\lambda = 0 \quad (\Psi = (2g)^{-1}(u - D)^2 + (\rho g)^{-1}p + Y). \tag{1.16}$$

This means that vorticity is conserved in the Lagrangian particle with the crossing of the discontinuity. The quantity Ψ is called the total head in hydraulics. The last relation means that the jump in the total head (calculated in a coordinate system that moves with the front) is independent of depth (since the pressure is hydrostatic).

It is known that vorticity is conserved along trajectories in the exact model of the plane-parallel flow of an ideal incompressible fluid. This fact supports the use of conservation law (1.14) in the theory of discontinuous flows — at least for low-amplitude jumps.

Along with Eqs. (1.15), we also need to satisfy the condition for the reduction in the total energy of the layer with the transition of the flow across the front when either $D - u \geq 0$ or $D - u \leq 0$ on the front. With allowance for (1.15), this condition can be represented in the form

$$\int_0^1 H(u - D)[2^{-1}\rho(u - D)^2 + p + \rho g\Phi]d\lambda \geqslant 0, \tag{1.17}$$

where $[f(x, t, \lambda)] = f(x(t) + 0, t, \lambda) - f(x(t) - 0, t, \lambda)$.

2. Properties of Relations for a Hydraulic Jump. Let us assume that the flow parameters $u = u_1$, $H = H_1$ are known ahead of the front along with the velocity D of the jump. We will examine the question of determining the parameters of the flow behind the front. The last relation of (1.15) is written as

$$[u_\lambda(u - D)] = 0.$$

Thus, $K = [(u - D)^2]$ is independent of λ. For the sake of definiteness, let the flow arrive at the front from the left: $u_1 - D > 0$. Then $[u - D)^2] = (u_1 - D)^2 - (u_2 - D)^2$ and

$$u_2 - D = ((u_1 - D)^2 - K)^{1/2} \ (K \leqslant K_* = \min_\lambda (u_1 - D)^2). \tag{2.1}$$

The first relation of (1.15) allows us to determine

$$H_2 = H_1(u_1 - D)((u_1 - D)^2 - K)^{-1/2} \tag{2.2}$$

(the subscript 2 pertains to parameters behind the front). Inserting u_2 and H_2 into the conservation law for total momentum, we obtain

$$F(K) - F(0) = 0 \tag{2.3}$$

to determine $K \in (0, K_*)$. Here

$$F(K) = \int_0^1 H_1(u_1 - D)((u_1 - D)^2 - K)^{1/2} d\lambda$$
$$+ 2^{-1} g\Big(\int_0^1 H_1(u_1 - D)((u_1 - D)^2 - K)^{-1/2} d\lambda\Big)^2.$$

By virtue of (1.16), the quantity $[2^{-1}\rho(u - D)^2 + p + \rho g\Phi] = \rho g[\Psi]$ is independent of λ. Thus, the sign of the expression in the left side of (1.17) coincides with the sign of $\sigma(K) = g[\Psi]$. The derivative of the function $\sigma(K)$ is calculated in the form

$$\sigma'(K) = 2^{-1}\Big(1 - g\int_0^1 H_1(u_1 - D)((u_1 - D)^2 - K)^{-3/2} d\lambda\Big).$$

At $K \to -\infty$ $\sigma'(K) \to 2^{-1}$, while at $K \to K_*$ $\sigma'(K) \to -\infty$. Since $\sigma''(K) < 0$, there exists a unique value K_0 such that $\sigma'(K_0) = 0$. It follows from the equality $\sigma(0) = 0$ that $\sigma(K)$ has a unique positive maximum at the point K_0. It follows from these properties that $\sigma(l) > 0$ for any l within the interval from 0 to K if $\sigma(K) \geq 0$.

The derivatives of $\sigma(K)$ and $F(K)$ are connected by the relation

$$F'(K) = -h_2(K)\sigma'(K)$$
$$\Big(h_2(K) = \int_0^1 H_1(u_1 - D)((u_1 - D)^2 - K)^{-1/2} d\lambda\Big). \tag{2.4}$$

We obtain the following equality as a corollary of this relation

$$F(K) - F(0) = -h_2(K)\sigma(K) + \int_0^K h'_2(l)\sigma(l)dl. \tag{2.5}$$

Equality (2.3) and the inequality $\sigma(K) \geq 0$ are incompatible at $K < 0$. In fact, if $\sigma(K) \geq 0$, then $\sigma(l) > 0$ for l between 0 and K. Also, since $h'_2(l) > 0$, both terms in the right side of (2.5) are negative at $K < 0$. Thus, roots K of (2.3) satisfying the inequality $\sigma(K) \geq 0$ can be found only on the interval $(0, K_*)$. The function σ will be positive at positive K only when the assigned parameters of the flow satisfy the inequality $\sigma'(0) > 0$; here, $K_0 \in (0, K_*)$. The function F decreases monotonically on the interval $(0, K_0)$ and reaches a minimum $F(K_0) < F(0)$. It then increases monotonically on the interval (K_0, K_*). If $F(K_*) > F(0)$, then Eq. (2.3) has the unique root K_s on the interval (K_0, K_*) and $\sigma'(K_s) < 0$ at point K_s.

At $F(K_*) \leq F(0)$, Eq. (2.3) has no roots on the interval $(0, K_*)$. If $K = K_*$, then $u_2 - D$ vanishes at those points where $(u_1 - D)^2 = K_*$. We will assume that the minimum value of $(u_1 - D)^2$ is reached only at the point $\lambda = \lambda_*$ We can then find the parameters u_2, Φ_2 behind the discontinuity, with Φ_2 undergoing a jump at $\lambda = \lambda_*$:

$$\Phi_2(x(t), t, \lambda) = \Phi_2^1(x(t), t, \lambda) + \eta(x(t), t)\theta(\lambda - \lambda_*). \tag{2.6}$$

Here, Φ_2^1 is a continuous function; $\theta(\lambda - \lambda_*)$ is the Heaviside function ($\theta = 1$ at $\lambda > \lambda_*$, $\theta = 0$ at $\lambda < \lambda_*$). In the regions $\lambda < \lambda_*$ and $\lambda > \lambda_*$, u_2 and H_2 are determined by formulas (2.1) and (2.2) with $K = K_*$. We take $u_2 = D$ in an interlayer of thickness η:

$$h_* = \int_0^{\lambda_*} H_1(u_1 - D)((u_1 - D)^2 - K_*)^{-1/2} d\lambda < Y < \eta + h_*.$$

The total depth of the layer of fluid behind the discontinuity is determined by the equality $h = h_2(K_*) + \eta$. We obtain the following for η from the conservation law for total horizontal momentum (1.15)

$$2^{-1}g(h_2(K_*) + \eta)^2 = 2^{-1}gh_2^2(K_*) - (F(K_*) - F(0)). \tag{2.7}$$

The right side of (2.7) is positive, since $F(K_*) < F(0)$. This allows us to unambiguously find the thickness of the interlayer

$$\eta = [h_2^2(K_*) - 2g^{-1}(F(K_*) - F(0))]^{1/2} - h_2(K_*). \tag{2.8}$$

By virtue of Eq. (2.4)

$$\sigma(K_*) = - h_2^{-1}(K_*)(F(K_*) - F(0)) - \int_0^{K_*} h_2^{-2}h_2'(l)(F(l) - F(0))dl > 0,$$

Inequality (1.17) is thus satisfied (when $y \in (h_*, h_* + \eta)u_2 = D$, with the interlayer making no contribution to the energy flux). We should note that the first condition of (1.15) is satisfied in the sense of generalized functions at values of Φ_2 determined by (2.6).

The inequalities $\sigma'(0) > 0$, $\sigma'(K_S) < 0$ can be represented in the form

$$1 - g\int_0^1 H_1(u_1 - D)^{-2}d\lambda > 0; \tag{2.9a}$$

$$1 - g\int_0^1 H_2(u_2 - D)^{-2}d\lambda < 0. \tag{2.9b}$$

It was shown in [6] that Eqs. (1.11) have real-valued characteristics determined by the differential equations $x'(t) = k^i(x, t)$ ($i = 1, 2$), where k^i are roots of the equation

$$1 - g\int_0^1 H(u - k)^{-2}d\lambda = 0 \quad (k^1 < \min_\lambda u,\ k^2 > \max_\lambda u). \tag{2.10}$$

Then inequalities (2.9) represent the conditions that must be satisfied for the flow to be supercritical ahead of the front and subcritical behind it. For the situation being examined, expressions (2.9) lead to the Lachs condition for the stability of a discontinuity $k_2^1 < D < k_1^1$ (the subscript denotes the state for which the characteristic root k^1 is calculated). Similar inequalities are satisfied when $F(K_*) < F(0)$. It follows from (2.9) that the velocity of the front D approaches the characteristic velocity as the amplitudes of the discontinuity approach zero.

After we find u_2, Φ_2, and $H_2 = \Phi_{2\lambda}$, we can change over to the variables X, Y, T once we have expressed λ in terms of X, Y, and T from the equation $Y = \Phi_2(X, T, \lambda)$ ($Y = \Phi_1(X, T, \lambda)$ in the region ahead of the jump). By virtue of these relations, the vertical Eulerian coordinate $Y = Y_2$ behind the jump is connected with the Eulerian coordinate ahead of the jump $Y = Y_1$ by the relation $Y_2 = Y_2(Y_1)$. We represent Eqs. (1.15) as

$$(u_1 - D)dY_1 = (u_2 - D)dY_2,\ (u_1 - D)du_1 = (u_2 - D)du_2,$$
$$\int_0^{h_1}(u_1 - D)^2dY_1 + 2^{-1}gh_1^2 = \int_0^{h_1}(u_2 - D)^2dY_2 + 2^{-1}gh_2^2.$$

In sum, we can conclude that two types of hydraulic jumps can propagate at supercritical (in the sense of (2.9a)) velocity D ahead of the front in the given state. For the first type, $u_2 - D \neq 0$ behind the front. This type of jump is possible if the assigned flow and the velocity of the jump D satisfy the conditions $F(K_*) - F(0) > 0$. If $F(K_*) - F(0) \leq 0$ and if $(u_1 - D)^2$ reaches a minimum at a single point, then a jump may occur with the presence of a stagnant (relative to the front) zone; on the front behind the jump, $u_2 - D = 0$ at $h_* \leq y \leq h_* + \eta$. If $(u_1 - D)^2$ reaches its minimum value immediately at several points, then only the total thickness of the interlayers can be determined from the balance of total momentum.

It should be noted that transients develop in steady (in the system of reference of the front) flows before the steady state is reached completely, and the formation of stagnant zones may depend on the previous history of the system. For a monotonic

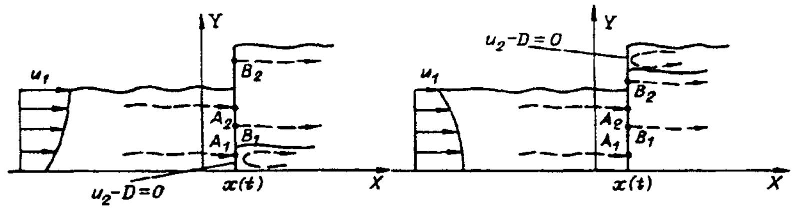

Fig. 2

velocity profile, when $u_{1Y} \neq 0$, the minimum value of $(u_1 - D)$ is reached either on the bottom or on the free surface. The formation of a stagnant section on the front near the free surface might be interpreted in one of two ways: the possible appearance of a pronounced "roll" [7]; the appearance of a region of reverse flow near the bottom. The authors of [3] presented examples of steady-state solutions of Eqs. (1.4) describing flows with closed streamlines and regions of reverse flow in the neighborhood of the critical layer $(u - D) = 0$. Possible flow patterns are shown in Fig. 2. It is clear that the pattern of flow in the region behind the front is also determined by conditions to the right of the front.

As an example, we will examine a steady hydraulic jump on a shear flow. Let a steady flow with the velocity $u_1(Y) = U\exp(\gamma Y)$, $v_1(Y) = 0$ (where U is a positive constant, $\gamma = \beta h_1^{-1}$, β = const) be assigned at $X < 0$ in a layer $0 \leq Y \leq h_1$. The law governing the correspondence between the entry and exit points of the trajectories $Y_2(Y_1)$ at the front is found as the solution of the problem

$$dY_2/dY_1 = u_1(Y_1)(u_1^2(Y_1) - K)^{-1/2},\ Y_2(0) = 0.$$

Integration yields the relation

$$\exp(\gamma Y_1) = \mathrm{ch}(\gamma Y_2) + (1 - K_1)^{1/2}\mathrm{sh}(\gamma Y_2)\ (K_1 = KU^{-2})$$

and the expression for velocity behind the front ($X > 0$)

$$u_2(Y_2) = U((1 - K_1)^{1/2}\mathrm{ch}(\gamma Y_2) + \mathrm{sh}(\gamma Y_2)). \tag{2.11}$$

The condition that must be satisfied for the flow ahead of the front to be supercritical (2.9a) is equivalent to the inequality

$$F_1^{-2} < 2\beta(1 - \exp(-2\beta))^{-1}\ (F_1 = U(gh_1)^{-1}).$$

The resulting relations allow us to calculate

$$\begin{aligned}F(K) - F(0) = (2\beta)^{-1}h_1U^2[e^{\beta}(e^{2\beta} - K_1)^{1/2} - (1 - K_1)^{1/2}\\ - K_1\ln\{(e^{\beta} + (e^{2\beta} - K_1)^{1/2})(1 + (1 - K_1)^{1/2})^{-1}\}\\ + \beta^{-1}F_1^{-2}(\ln\{(e^{\beta} + (e^{2\beta} - K_1)^{1/2})(1 + (1 - K_1)^{1/2})^{-1}\})^2 - e^{2\beta} + 1 - \beta F_1^{-2}].\end{aligned}$$

At $\beta > 0$, $K_1 = 1$ corresponds to the value $K = K_* = U^2$. Let $\beta = \ln 3$. The inequality $F(K_*) - F(0) > 0$ is satisfied for Froude numbers F_1 satisfying the inequalities $1.17468 > F_1 > 0.63604$. In this case, the equation $F(K) - F(0) = 0$ has the single root K_1 in the interval (0.1). Then $v_2(Y) = 0$ and $u_2(Y)$ is determined by Eq. (2.11) in the region behind the front ($X > 0$), while the depth of the fluid layer is determined by the formula

$$h_2 = h_2(K_1) = \beta^{-1}h_1\ln\{(e^{\beta} + (e^{2\beta} - K_1)^{1/2})(1 + (1 - K_1)^{1/2})^{-1}\}\ (\beta = \ln 3).$$

If $F_1 > 1.17468$, then the total depth of the fluid layer behind the jump is determined by the expression

$$h_2 = [h_2^2(1) - 2g^{-1}(F(U^2) - F(0))]^{1/2}.$$

At $0 < Y < h_2 - h_2(1)$, $u_2(y) = 0$ and $v_2(Y) = 0$. In the layer $h_2 - h_2(1) < Y < h_2$, the horizontal component of velocity $u_2(Y)$ is given by Eq. (2.11). Here, Y_2 must be replaced by the quantity $Y + h_2(1) - h_2$. The vertical component of velocity $v_2(Y) = 0$. The second solution describes flow with a hydraulic jump and a stagnant zone near the bottom.

Let us address the question of determining the flow parameters $u_2(x, t, \lambda)$, $H_2(x, t, \lambda)$ behind the front of a weak hydraulic jump on the basis of assigned parameters $u_1(x, t, \lambda)$, $H_1(x, t, \lambda)$, D. Here, it is convenient to take $K = \varepsilon << 1$. If we represent $u_2 - D$ and H_2 in the form

$$u_2 - D = u_1 - D + \varepsilon\delta u_2,\ H_2 = H_1 + \varepsilon\delta H_2$$

(δu_2 and δH_2 are the sought perturbations) and if we satisfy the relations for the discontinuity with first-order accuracy, we will have

$$\delta u_2 = -(2(u_1 - D))^{-1},\ \delta H_2 = 2^{-1}H_1(u_1 - D)^{-2}. \tag{2.12}$$

Here, it is necessary that $D = k_1^1$, where k_1^1 is the root of characteristic equation (2.10) calculated for state 1. In Eulerian coordinates, we obtain the following representations for the parameters of the flow behind the jump:

$$u_2(Y) = u_1(Y) - 2^{-1}\varepsilon((u_1(Y) - D)^{-1} + u_{1Y}(Y))\int_0^Y \frac{d\eta}{(u_1(\eta) - D)^2},$$
$$h_2 = h_1 + (2g)^{-1}\varepsilon.$$

The correspondence between the points on the jump is given by the equation:

$$Y_2 = Y_1 + \frac{\varepsilon}{2}\int_0^{Y_1}\frac{d\eta}{(u_1(\eta) - D)^2}.$$

It was shown in [6] that, under certain conditions (conditions under which the system is hyperbolic), system (1.11) leads to the Riemannian invariants

$$R_t + uR_x = 0,\ \omega_t + u\omega_x = 0,\ r_{it} + k^i r_{ix} = 0\ (i = 1, 2).$$

Here, $R(x, t, \lambda)$, $\omega(x, t, \lambda)$, and $r_i(x, t)$ are determined by the formulas

$$R = u(\lambda) - g\int_0^1 \frac{H(\nu)d\nu}{u(\nu) - u(\lambda)},\ \omega = u_\lambda H^{-1},\ r_i = k^i - g\int_0^1\frac{H(\nu)d\nu}{u(\nu) - k^i}$$

(k^1 and k^2 are roots of the characteristic equation; for the sake of brevity, the dependence of the functions on the variables x and t is not indicated in the notation). The author of [8] analyzed simple waves (particular solutions of system (1.11) having the form $u = u(\alpha(x, t), \lambda)$, $H = H(\alpha(x, t), \lambda)$, $\alpha_t + k^1\alpha_x = 0$) and showed that $R = R(\lambda)$, $\omega = \omega(\lambda)$, $r_2 = \overset{0}{r}_2 = \text{const}$, in the region of a simple wave if $R = R(\lambda)$, $\omega = \omega(\lambda)$, $r_i = \text{const}$ ahead of this wave (conservation of the Riemannian invariants R, ω, and r_2 in a simple wave propagating at the characteristic velocity k^1).

For the given case of flow behind a weak hydraulic jump

$$\delta R_2 = \delta u_2 - g\int_0^1\frac{\delta H_2(\nu)d\nu}{u_1(\nu) - u_1(\lambda)} + g\int_0^1\frac{H_1(\nu)(\delta u_2(\nu) - \delta u_2(\lambda))}{(u_1(\nu) - u_1(\lambda))^2}d\nu = 0,$$
$$\delta\omega = (\delta u_2)_\lambda H_1^{-1} - u_{1\lambda}H_1^{-2}\delta H_2 = 0, \tag{2.13}$$
$$\delta r_2 = -g\int_0^1\frac{\delta H_2(\nu)d\nu}{u_1(\nu) - k_1^2} + g\int_0^1\frac{H_1(\nu)\delta u_2 d\nu}{(u_1(\nu) - k_1^2)^2} = 0.$$

Equations (2.13) are verified directly using Eqs. (2.12) and Eq. (2.10), the roots of which are k_1^1 and k_1^2. Thus, in the transition through a weak leftward hydraulic jump, the changes in the Riemannian invariants R, ω, and r_2 are of a higher order than the amplitude of the jump. Here, as in gasdynamics, there is an analogy between the behavior of simple waves and low-amplitude discontinuities.

The model proposed here for hydraulic jumps describes a new type of hydrodynamic discontinuity associated with sudden changes not only in velocity and pressure, but also the Eulerian coordinate of particles. In an actual flow, the movement of a particle to a new elevation occurs over a finite distance. However, the size of the region over which this transition takes place is small compared to the characteristic scale of long-wave processes. Thus, in an approximate theory, the transition is modeled by a discontinuity in the particle trajectory. An analysis of the model shows that this approach is quite suitable for describing low-amplitude discontinuities. Within the range of large amplitudes, the model indicates that changes might take place in the structure of the flow within the region behind the jump. Empirical velocity profiles measured in the depth direction are needed to determine the level of agreement between theory and experiment.

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